FREE BOUNDARY REGULARITY IN PROBLEMS GOVERNED BY DEGENERATE OPERATORS

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1. The research project

1.1. **State of the play.** The research in free boundary problems obtained a major impulse since the introduction of the geometric approach mainly due to W. Alt, L. Caffarelli and A. Friedman [4, 5, 6, 7, 8, 9, 10] and successively improved by Luis Caffarelli himself in [11, 12, 13], see also [14] for a wider bibliography about that subject.

The first results about the regularity of free boundary of the solutions have been obtained in free boundary problems governed by linear operators, [15, 28] and then extended to cases governed by nonlinear operators in non-divergence form, see e.g. [30, 31, 36, 37, 38, 24]. Those results have been extended to nonlinear operators in divergence form, essentially governed by the p-Laplace operator, see [32, 33, 34]. In all the cited cases, the problems were homogeneous.

With the contribution of D. De Silva, [16], the research interest moved to the non-homogeneous case, because the technique introduced in [16] immediately appeared as very flexible to be adapted to several operators [17, 18, 19, 34, 20]. During the last years, the interest focused on operators that might be, for different reasons degenerate. Let us think, for instance to the $p-$Laplace operator in a non-homogeneous one phase case.

1.2. **Some free boundary problems. Elliptic two-phase problem.**

One of the simplest problem, in two phase case, can be stated as follows. Let $u \in C(\Omega, \mathbb{R})$ be a viscosity solution to the problem

$$(1)\qquad \begin{cases} \Delta u = f, & \Omega^+(u) := \{x \in \Omega : \ u > 0\}, \\ \Delta u = f, & \Omega^-(u) := \{x \in \Omega : \ u \leq 0\}^o, \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 1, & \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega, \end{cases}$$

where $f \in C(\Omega, \mathbb{R}) \cap L^\infty(\Omega)$, in $\Omega \subset \mathbb{R}^n$, is an assigned function and ν, formally, denotes the unit normal vector on $\mathcal{F}(u)$ at the points belonging to $\mathcal{F}(u)$, with the convention that ν is pointing respectively inside to the set $\Omega^+(u)$ for u^+ as well as the same happens for $\Omega^-(u)$ for u^-.

The set $\mathcal{F}(u)$ is the so called *free boundary* set in the problem, because it is an unknown, even if on such set $\mathcal{F}(u)$ a condition is assumed. More precisely, on such unknown set, the gradient jump condition $(u_\nu^+)^2 - (u_\nu^-)^2 = 1$ has to be satisfied.

One of the most important questions concerns the regularity of the free boundary set, that is the regularity of $\mathcal{F}(u)$.

Stefan problem

Another well known free boundary problem may be stated in the parabolic version and it represents the companion of the two-phase problem and is well known in literature as the Stefan problem.

Let $u \in C(\Omega \times (a,b))$, $\Omega \subset \mathbb{R}^n$ be a viscosity solution to

$$
\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \Delta u = 0, \quad \Omega^+(u) := \{(x,t) \in \Omega \times (a,b): \ u > 0\}, \\ \frac{\partial u}{\partial t} - \Delta u = 0, \quad \Omega^-(u) := \{(x,t) \in \Omega \times (a,b): \ u \leq 0\}^o, \\ -V_\nu = (u_\nu^+)^2 - (u_\nu^-)^2, \quad \mathcal{F}(u) := \partial\Omega^+(u) \cap (\Omega \times (a,b). \end{array} \right. \tag{2}
$$

Where $V_\nu := \frac{\frac{\partial u^+}{\partial t}}{u_\nu^+}$ denotes the speed of the free boundary along the spatial normal direction ν.

Even in this case, the regularity of the free boundary represents one of the main information that we would like to know.

1.3. **Proposals.** The first goal of this research project is based on the idea that a solution to a one or two phase free boundary problems should have a free boundary endowed with regularity much higher than one might expect. In fact, one of the solutions satisfies not only a PDE equation but, in addition, has to satisfy the condition assumed on the free boundary. These two facts force a better regularity of the solution and of the free boundary itself. In [18, 19, 20] regularity results of (1) have been proved, as well as free boundary regularity for (2) have been obtained in [1, 2, 3], see also [29].

With this research we would like to attack a couple of problems that naturally arise. The first one concerns with the regularity of the free boundary when the leading operator may be degenerate. For instance, in the case of the Kohn-Laplace operator in the Heisenberg group $\mathbb{H}^1$, the problem (1) becomes, as it has been proved in [23]:

$$
\left\{ \begin{array}{l} \Delta_{\mathbb{H}^1} u = f, \quad \Omega^+(u) := \{x \in \Omega: \ u > 0\}, \\ \Delta_{\mathbb{H}^1} u = f, \quad \Omega^-(u) := \{x \in \Omega: \ u \leq 0\}^o, \\ \|\nabla_{\mathbb{H}^1} u^+\|^2 - \|\nabla_{\mathbb{H}^1} u^+\|^2 = 1, \quad \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega. \end{array} \right. \tag{3}
$$

Here $\Omega \subset \mathbb{H}^1$ is an open set and $\mathbb{H}^1$ denotes the simplest Heisenberg group where

$$
\Delta_{\mathbb{H}^1} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + 2y\frac{\partial^2}{\partial t \partial x} - 2x\frac{\partial^2}{\partial t \partial y} + 4(x^2+y^2)\frac{\partial^2}{\partial t^2},
$$

as well as $\nabla_{\mathbb{H}^1} u = XuX + YuY \equiv (Xu, Yu)$ and $(x,y,t) \in \mathbb{H}^1 \equiv \mathbb{R}^3$ so that $\|\nabla_{\mathbb{H}^1} u\| = \sqrt{(Xu)^2 + (Yu)^2}$.

It is worth to point out that the matrix associated with $\Delta_{\mathbb{H}^1}$ is non-negative definite and in particular its smallest eigenvalue is always zero. So that the problem is very degenerate.

In the non-degenerate case like (1), one of the main tools for knowing the regularity of the free boundary is based on the existence of a monotonicity formula, [6]. Concerning (3), only some partial results that deal with existence/non-existence of a monotonicity formula have been obtained in [25]. So, one of the tasks of this project is to improve the knowledge concerning the existence or non-existence of a monotonicity formula for this type of problem. In addition, we want to verify if the strategy described in [16] may be applied even to this degenerate situation, in particular concerning what type of regularity of the free boundary of viscosity solutions to (3) can be supposed satisfied.

Concerning the Stefan problem, we point out that the heat operator itself in the Euclidean space time may be regarded as a degenerate elliptic operator and some results have been obtained in [21] in the non-homogeneous case considering one phase case. The one phase case requires that the condition on the free boundary reduces to $-V_\nu = (u_\nu^+)^2$. Hence we want to extend the one-phase results to the two-phase case in adapting the technique tested in [21].

In addition, we want to explore a different approach recently introduced by De Silva and Savin, [22], mainly focused on the quasi-minima of linear operators. We would like to check if those ideas may be extended to the problem (3) in the one phase setting. Moreover, we are also interested in dealing with the problem in the one phase case for the $p-$Laplace operator, but in the Euclidean frame. In the long run, we hope to generalize these techniques to operators with non-standard growth, like the $p(x)-$Laplace operator, [26, 27], always in the Euclidean case.

1.4. **Activities.** The hired researcher will interact with me and possibly with other members of the Mathematics Department of Bologna on the project. It has been scheduled in 2022 the visits of some experts in free boundary problems in the frame of the activities planned at the University of Bologna thanks to the funds available in *The interplay of Geometry, Combinatorics and Representation Theory* that is part of *Bando strutture, "Promozione di iniziative innovative nell'ambito degli accordi quadro" 2020.* During those visits the hosts will collaborate with me, the colleagues in the Department and the researcher who will work on this project. In addition, concerning the quasi minima approach to the $p-$Laplace operator, the hired researcher will have to collaborate with me, Enrico Valdinoci and Serena Dipierro of The University of Western Australia.

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